
Numerical Techniques (others): Paper ICA2016-736**Passive time-domain numerical designs for room acoustics simulation****Stefan Bilbao^(a), Brian Hamilton^(b)**^(a)Acoustics and Audio Group, University of Edinburgh, United Kingdom,
sbilbao@staffmail.ed.ac.uk^(b)Acoustics and Audio Group, University of Edinburgh, United Kingdom,
brian.hamilton@ed.ac.uk**Abstract**

The design of stable time domain numerical simulation methods for room acoustics simulation is a challenging problem. One chief difficulty is in the determination of appropriate stable boundary terminations, particularly when the room geometry is irregular, and when the wall condition is spatially-varying and/or frequency-dependent in a non-trivial way. In this paper, design strategies for stable simulation are presented, based on the finite volume time domain method (FVTD), which, due to its unstructured character, allows for flexible modelling of irregular room geometries. Furthermore, FVTD reduces to the popular finite difference time domain (FDTD) method under certain choices of regular structured mesh. Under locally-reactive wall conditions, the boundary condition can be characterised by a positive real admittance function, variable over the extent of the room boundary. Using frequency-domain analysis techniques, it can be shown that solutions to the complete system are non-increasing. Furthermore, such analysis techniques can be extended to the case of discrete time simulations, leading to numerical stability conditions for a complete room simulation. Distinct explicit and implicit time-domain simulation methods are analysed in this manner. Extensions to the case of non-locally reactive conditions are discussed.

Keywords: room acoustics; finite difference time domain; finite volume time domain

Passive time-domain numerical designs for room acoustics simulation

1 Introduction

Wave-based time domain methods, such as finite difference (FDTD) [1] and finite volume (FVTD) [2, 3] methods allow, in theory, for complete solutions to the problem of room acoustics simulation. Despite their high computational cost, modern computer hardware allows for audio rate simulation, for large acoustic spaces, in a reasonable amount of time.

A major hurdle, in the design of such schemes, has been the incorporation of room geometries and wall conditions of realistic type, including spatial variation and frequency dependence of wall conditions; such features are unavoidable attributes of any real world application. Though there exist various techniques for the termination of FDTD schemes under such conditions, determining conditions for numerical stability can be challenging, and is not often attempted in the literature. One avenue of approach is through the use of FVTD methods, which generalise certain well-known FDTD methods defined over regular grids. When accompanied by energy analysis, sufficient stability conditions may be found, provided that the wall conditions are defined in terms of passive admittances, and that discretisation of the wall admittances preserves this feature. In short, stability analysis for the problem as a whole may be divided into an analysis of the numerical method over the interior, under perfectly reflective (Neumann) conditions, and an additional analysis of the numerical boundary conditions. To date, certain classes of locally-reactive wall admittances have been approached in this way, where passivity is ensured through circuit representations with positive element values [4].

In this paper, a general frequency-domain stability analysis technique is presented, suitable for FVTD or certain FDTD methods, and for which the wall admittance is characterised in terms of general positive real functions, perhaps lacking a concrete circuit representation. In Section 2, a standard model of wave propagation is presented, accompanied by a boundary conditions framed in terms of positive real admittances, variable over the room boundary. In Section 3, after finite volume discretisation, frequency-domain analysis of the semi-discrete system (a system of ordinary differential equations) is carried out, and the positive realness property is employed directly to show that all solutions are exponentially damped. In Section 4, this analysis is extended to the case of fully discrete schemes of various type, in order to arrive at stability conditions which may be deduced purely from the Neumann problem, without any interaction with numerical boundary conditions. Further comments, and the extension of this analysis to the case of non-locally reactive conditions are briefly discussed in Section 5.

2 Room Acoustics: Model System

Consider a room geometry $\mathcal{D} \subset \mathbb{R}^3$, with boundary $\partial\mathcal{D}$, assumed smooth. A good approximation to the dynamics of the acoustic field within the enclosure is given in terms of the first order

system:

$$\rho \partial_t \mathbf{v} + \nabla p = \mathbf{0} \quad \frac{1}{\rho c^2} \partial_t p + \nabla \cdot \mathbf{v} = 0 \quad (1)$$

Here, $p(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$ represent, respectively, the pressure deviation from atmospheric in Pa, and the vector particle velocity in m/s, at coordinate $\mathbf{x} \in \mathcal{D}$, and for time $t \geq 0$. ρ and c represent, respectively, the density of air in kg/m³ and the wave speed in m/s. ∂_t represents partial differentiation with respect to time t , and ∇ and $\nabla \cdot$ represent the gradient and divergence operations, respectively, in three spatial dimensions. In this short contribution, only the initial value problem will be considered, and not sources, so additional data necessary are the initial conditions $p(\mathbf{x}, 0)$ and $\mathbf{v}(\mathbf{x}, 0)$. System (1) results from the linearisation of the Navier Stokes equations and the subsequent neglect of viscothermal losses. Such losses can indeed be reincorporated into the system above, and have a non-negligible effect on decay times, particularly in the high-frequency range [5]. For simplicity however, they are not considered here.

A second order form corresponding to (1) is the usual starting point for wave-based numerical simulations:

$$\partial_t^2 \Psi = c^2 \Delta \Psi \quad \text{where} \quad p = \rho \partial_t \Psi \quad \mathbf{v} = -\nabla \Psi \quad (2)$$

It is written here in terms of a scalar velocity potential [6] $\Psi(\mathbf{x}, t)$, using the Laplacian operator $\Delta = \nabla \cdot \nabla$. While it is the first order system (1) which will be considered henceforth in this article, it should be borne in mind that numerical implementations, including those presented here, are most efficient when written in terms of the second order system (2).

2.1 Energy Balance

An energy balance follows for system (1):

$$\frac{d}{dt} \xi_i + \mathfrak{B} = 0 \quad \xi_i = \iiint_{\mathcal{D}} \frac{\rho}{2} |\mathbf{v}|^2 + \frac{p^2}{2\rho c^2} dV \quad \mathfrak{B} = \iint_{\partial \mathcal{D}} p v_n d\sigma \quad (3)$$

Here, the scalar function $\xi = \xi_i(t) \geq 0$ is the energy stored in the acoustic field over the room interior, and the scalar function $\mathfrak{B} = \mathfrak{B}(t)$ represents the total instantaneous power transfer to the room boundary. $v_n = \mathbf{n} \cdot \mathbf{v}$ is the outward normal component of the particle velocity at the room boundary, where \mathbf{n} is the outward normal vector over $\partial \mathcal{D}$. A physically realistic condition is that the wall condition be passive, so that one may write

$$\mathfrak{B} = \frac{d}{dt} \xi_b + \mathfrak{Q} \quad \xi_b \geq 0 \quad \mathfrak{Q} \geq 0 \quad (4)$$

where $\xi_b = \xi_b(t)$ represents energy stored at the wall, and $\mathfrak{Q} = \mathfrak{Q}(t)$ represents instantaneous power loss over the wall. A general energy balance holds:

$$\frac{d}{dt} \xi + \mathfrak{Q} = 0 \quad \xi = \xi_i + \xi_b \geq 0 \quad (5)$$

Under these conditions, the total energy stored in the acoustic field and over the room boundary must be monotonically non-increasing over time.

2.2 Locally-reactive Wall Conditions

A general locally-reactive wall condition, at coordinate $\mathbf{x}_0 \in \partial\mathcal{D}$, may be written as a differential relationship between $p(\mathbf{x}_0, t)$ and $v_n(\mathbf{x}_0, t)$. For practical applications, it is relationships of finite order M which are of interest:

$$\sum_{q=0}^M \beta_q \frac{d^q}{dt^q} p = \sum_{q=0}^M \alpha_q \frac{d^q}{dt^q} v_n \quad (6)$$

where here, $\alpha_q(\mathbf{x}_0)$, $\beta_q(\mathbf{x}_0)$, $q = 0, \dots, M$, and with $\alpha_0 = 1$, are coefficients which depend on the coordinate \mathbf{x}_0 , and represent the local properties of the wall material.

In general, for given coefficients α_q and β_q , $q = 0, \dots, M$ it can be difficult to determine explicit forms for the stored energy ξ_b and power loss \mathcal{Q} at the wall. An equivalent criterion follows from the frequency-domain, or Laplace-transformed form of (6):

$$\hat{v}_n(\mathbf{x}_0, s) = Y(\mathbf{x}_0, s) \hat{p}(\mathbf{x}_0, s) \quad Y = \frac{\sum_{q=0}^M \beta_q s^q}{\sum_{q=0}^M \alpha_q s^q} \quad (7)$$

Here, s is the transform variable, or complex frequency $s = j\omega + \sigma$, and $\hat{v}_n(\mathbf{x}_0, s)$ and $\hat{p}(\mathbf{x}_0, s)$ are the Laplace transforms of $v_n(\mathbf{x}_0, t)$ and $p(\mathbf{x}_0, t)$. $Y(\mathbf{x}_0, s)$ is the wall admittance at \mathbf{x}_0 , and a condition for passivity is that Y is locally positive real [7]:

$$\operatorname{Re}(Y(\mathbf{x}_0, s)) \geq 0 \quad \text{when} \quad \operatorname{Re}(s) \geq 0 \quad (8)$$

A well-known result is that for a positive real function of finite order, there corresponds at least one realisation, which can be written using the acoustical/electrical circuit analogy, as a one-port element composed of resistors, capacitors and inductors of positive element values [7]. Once a realisation is available, one may proceed directly to explicit expressions for stored energy ξ_b and \mathcal{Q} in terms of the reactive and resistive components, respectively. Previous work by these authors on the problem of passive termination of room acoustics simulation algorithms has relied on specialised forms of the admittance Y admitting a simple realisation (such as a decomposition of Y into a realisation consisting of a parallel set of RLC branches) [4], without having to resort to general realisations such as the Brune synthesis [8].

In the remainder of this article, we illustrate the means whereby passive and thus stable numerical methods may be constructed, under general frequency-dependent boundary conditions, using only the property of positive realness, or passivity of the wall admittance, and not any particular realisation. The family of wall conditions which may be modelled is thus extended considerably beyond that considered in previous work.

3 Semi-discrete Forms: Finite Volume Methods

If one is interested in designing a full time-domain wave-based simulation method for system (1), that operates as a recursion over the entire room volume (unlike, e.g., boundary element techniques), a first step is the spatial discretisation of the problem. A standard technique is the finite difference time domain method (FDTD), where the spatial discretisation is usually taken

to be regular (according, e.g., to a Cartesian grid). A different approach, allowing for much more accurate modeling of boundary conditions, and a means of performing stability analysis, is through the use of finite volume time domain (FVTD) methods, generally written over an unstructured collection of cells. Under a regular arrangement of cells over the problem interior, FVTD schemes reduce to certain well-known families of FDTD schemes, while allowing for fitting of unstructured cells at the room boundary, and incur little to no additional computational expense over FDTD methods. See [2, 3] for more on the basics of FVTD methods.

For a given problem domain $\mathcal{D} \subset \mathbb{R}^3$, suppose that it is divided up into N non-overlapping polyhedral cells Ω_j , $j = 1, \dots, N$, with associated volumes $V_j = |\Omega_j|$. The faces of the polyhedra may be divided into two categories: the first category consists of those faces which separate two cells, and which will be called internal faces here, labelled σ_e , $e = 1, \dots, N_e$, where N_e is the number of such internal faces. The area of such a face will be named $S_e = |\sigma_e|$. A given internal face σ_e is associated with an inter-cell distance H_e between the neighbouring cells. The second category consists of those faces which lie on the boundary of the domain \mathcal{D} , and which will be called boundary faces here, labelled σ'_b , $b = 1, \dots, N_b$, where N_b is the number of such boundary or external faces. Again, $S'_b = |\sigma'_b|$ denotes the area of the face σ'_b .

It is useful to express the adjacency of cells between one another and to the boundary through graphs. First, for the case of inter-cell adjacency, consider an arbitrary internal face σ_e , $e = 1, \dots, N_e$. Such a face separates two adjacent cells $\Omega_{j_e^+}$ and $\Omega_{j_e^-}$, where $j_e^+, j_e^- \in \{1, \dots, N\}$, and $j_e^+ > j_e^-$. Define an oriented adjacency tensor q_{ej} , $e = 1, \dots, N_e$, $j = 1, \dots, N$ by

$$q_{ej_e^+} = 1 \quad q_{ej_e^-} = -1 \quad q_{ej} = 0 \quad \text{when } j \neq j_e^+, j_e^- \quad (9)$$

Similarly, for the case of boundary cell adjacency, consider a boundary face σ'_b , $b \in \{1, \dots, N_b\}$. The boundary face corresponds to one cell Ω_{j_b} . Define a non-oriented adjacency tensor q'_{bj} , $b = 1, \dots, N_b$, $j = 1, \dots, N$, as

$$q'_{bj_b} = 1 \quad q'_{bj} = 0 \quad \text{when } j \neq j_b \quad (10)$$

In order to approximate the system (1) using finite volume methods, first associate with each cell Ω_j a pressure variable $p_j(t)$, $j = 1, \dots, N$, representing an average pressure over the cell, and with each internal face σ_e , an averaged scalar velocity component v_e , $e = 1, \dots, N_e$ oriented normal to σ_e , and in the direction from cell $\Omega_{j_e^-}$ to $\Omega_{j_e^+}$. Furthermore, define scalar velocity components v'_b , $b = 1, \dots, N_b$, oriented outward to the boundary faces σ'_b .

A finite volume discretisation of (1) may be written as

$$\rho H_e \frac{d}{dt} v_e + \sum_{j=1}^N q_{ej} p_j = 0 \quad \frac{V_j}{\rho c^2} \frac{d}{dt} p_j - \sum_{e=1}^{N_e} q_{ej} S_e v_e + \sum_{b=1}^{N_b} q'_{bj} S'_b v'_b = 0 \quad (11)$$

The first of (11) above results from the approximation of the gradient of p normal to the face σ_e ; the second results from the integration of the second of (1) over the cell Ω_j and the application of the divergence theorem. Note that, as yet, the system is not closed, as the values for the boundary normal velocities v'_b have not been set.

3.1 Laplace Transformation and Boundary Conditions

Under a Laplace transformation, the ODE system given in (11) may be written as

$$\rho H_e s \hat{v}_e + \sum_{j=1}^N q_{ej} \hat{p}_j = \rho H_e v_e(0) \quad \frac{V_j}{\rho c^2} s \hat{p}_j - \sum_{e=1}^{N_e} q_{ej} S_e \hat{v}_e + \sum_{b=1}^{N_b} q'_{bj} S'_b \hat{v}'_b = \frac{V_j}{\rho c^2} p_j(0) \quad (12)$$

where transform pairs may be written as

$$p_j(t) \rightarrow \hat{p}_j(s) \quad v_e(t) \rightarrow \hat{v}_e(s) \quad v'_b(t) \rightarrow \hat{v}'_b(s) \quad (13)$$

in terms of the transform variable s . Initial conditions for p_j and v_e have appeared in (12).

A general M_b th order rational approximation to a locally reactive boundary condition, at boundary face σ'_b , can be written as a differential relationship between the outward normal velocity at the face v'_b , and the pressure p'_b at the unique associated cell, as

$$S'_b \sum_{q=0}^{M_b} \alpha_{b,q} \frac{d^q}{dt^q} v'_b = \sum_{q=0}^{M_b} \beta_{b,q} \frac{d^q}{dt^q} p'_b \quad \text{where} \quad p'_b = \sum_{j'=1}^N q'_{bj'} p_{j'} \quad (14)$$

in terms of the constants $\alpha_{b,q}$ and $\beta_{b,q}$. When Laplace transformed, this leads to the relationship

$$S'_b \hat{v}'_b = Y_b \sum_{j'=1}^N q'_{bj'} \hat{p}_{j'} + f_b(0) \quad Y_b = \frac{\sum_{q=0}^{M_b} \beta_{b,q} s^q}{\sum_{q=0}^{M_b} \alpha_{b,q} s^q} \quad (15)$$

where, for passivity, $Y_b(s)$ is positive real, and represents the wall admittance over face σ'_b , and where $f_b(0)$ is the result of the various initial conditions needed to complete the description (14). When (12) and (15) are combined, a complete system results:

$$\rho H_e s \hat{v}_e + \sum_{j=1}^N q_{ej} \hat{p}_j = \rho H_e v_e(0) \quad \frac{V_j}{\rho c^2} s \hat{p}_j - \sum_{e=1}^{N_e} q_{ej} S_e \hat{v}_e + \sum_{b=1}^{N_b} q'_{bj} Y_b \sum_{j'=1}^N q'_{bj'} \hat{p}_{j'} = \frac{V_j}{\rho c^2} p_j(0) + \sum_{b=1}^{N_b} q'_{bj} f_b(0) \quad (16)$$

3.2 Vector-matrix Form and Scaling

It is useful, for analysis purposes, to write (16) in a vector-matrix form. To this end, let $\hat{\mathbf{p}} = [\hat{p}_1, \dots, \hat{p}_N]^T$ and $\hat{\mathbf{v}} = [\hat{v}_1, \dots, \hat{v}_{N_e}]^T$, to give

$$s \rho \mathbf{H} \hat{\mathbf{v}} + \mathbf{Q} \hat{\mathbf{p}} = \mathbf{f}_v \quad \frac{s}{\rho c^2} \mathbf{V} \hat{\mathbf{p}} - \mathbf{Q}^T \mathbf{S} \hat{\mathbf{v}} + \mathbf{Q}'^T \mathbf{Y} \mathbf{Q}' \hat{\mathbf{p}} = \mathbf{f}_p \quad (17)$$

Here, \mathbf{H} is an $N_e \times N_e$ diagonal matrix, with H_e , $e = 1, \dots, N_e$ on the diagonal, \mathbf{V} is an $N \times N$ diagonal matrix, with V_j , $j = 1, \dots, N$ on the diagonal, and \mathbf{S} is an $N_e \times N_e$ diagonal matrix, with S_e , $e = 1, \dots, N_e$ on the diagonal. All are positive definite. \mathbf{Q} and \mathbf{Q}' are matrix representations of the tensors q_{ej} and q_{bj} , and are of size $N_e \times N$ and $N_b \times N$, respectively. \mathbf{f}_v and \mathbf{f}_p are $N_e \times 1$ and $N \times 1$ vectors including all initial conditions. Finally, \mathbf{Y} is an $N_b \times N_b$ matrix, with the admittances Y_b , $b = 1, \dots, N_b$ on the diagonal.

Introduce the scaled variables $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{v}}$, and the scaled admittance matrix $\tilde{\mathbf{Y}}$, defined as

$$\tilde{\mathbf{p}} = \frac{1}{\rho^{1/2}c} \mathbf{V}^{1/2} \hat{\mathbf{p}} \quad \tilde{\mathbf{v}} = \rho^{1/2} \mathbf{H}^{1/2} \mathbf{S}^{1/2} \hat{\mathbf{v}} \quad \tilde{\mathbf{Y}} = \rho c \mathbf{Y} \quad (18)$$

The system as a whole may now be written, after removing the tilde notation, as

$$\underbrace{\begin{bmatrix} s\mathbf{I}_N + \mathbf{B}\mathbf{Y}(s)\mathbf{B}^T & -\mathbf{A}^T \\ \mathbf{A} & s\mathbf{I}_{N_e} \end{bmatrix}}_{\mathbf{M}(s)} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_p(0) \\ \mathbf{f}_v(0) \end{bmatrix} \quad (19)$$

where

$$\mathbf{A} = c\mathbf{H}^{-1/2}\mathbf{S}^{1/2}\mathbf{Q}\mathbf{V}^{-1/2} \quad \mathbf{B} = c^{1/2}\mathbf{Q}'\mathbf{V}^{-1/2} \quad \tilde{\mathbf{f}}_p = \rho^{1/2}c\mathbf{V}^{-1/2}\mathbf{f}_p \quad \tilde{\mathbf{f}}_v = \rho^{-1/2}\mathbf{H}^{1/2}\mathbf{S}^{-1/2}\mathbf{f}_v \quad (20)$$

and where \mathbf{I}_N and \mathbf{I}_{N_e} are identity matrices of size $N \times N$ and $N_e \times N_e$, respectively.

3.3 Solutions

The behaviour of solutions to (19) is determined by the matrix $\mathbf{M}(s)$. If the zeros of $\mathbf{M}(s)$ are located in the left half plane (i.e., for $\text{Re}(s) \leq 0$), then no solution can exhibit exponential growth. To show that this is the case, assume that $\mathbf{M}(s)$ does possess a zero for $s = s^*$, with $\text{Re}(s^*) > 0$. Then $\det(\mathbf{M}(s^*)) = 0$. But, taking the Schur complement $\bar{\mathbf{M}}$ of the lower diagonal block $s\mathbf{I}_{N_e}$ leads to the condition

$$\det \left(\underbrace{s^* \mathbf{I}_N + \mathbf{B}\mathbf{Y}(s^*)\mathbf{B}^T + \frac{1}{s^*} \mathbf{A}^T \mathbf{A}}_{\bar{\mathbf{M}}(s^*)} \right) = 0 \quad (21)$$

which implies that there must exist a vector \mathbf{x} for which $\mathbf{x}^\dagger \bar{\mathbf{M}}(s^*) \mathbf{x} = 0$, where \mathbf{x}^\dagger is the conjugate transpose of the vector \mathbf{x} . But, defining $\mathbf{w} = \mathbf{B}^T \mathbf{x}$ one may write

$$\text{Re} \left(\mathbf{x}^\dagger \bar{\mathbf{M}}(s^*) \mathbf{x} \right) = \text{Re}(s^*) |\mathbf{x}|^2 + \frac{\text{Re}(s^*)}{|s^*|^2} |\mathbf{A}\mathbf{x}|^2 + \sum_{b=1}^{N_b} \text{Re}(Y_b(s^*)) |w_b|^2 = 0 \quad (22)$$

By the positive realness property of Y_b , $\text{Re}(Y_b(s^*)) \geq 0$, and thus the equation above cannot be satisfied for $\text{Re}(s^*) > 0$, leading to a contradiction.

4 Discrete-time Forms

The Laplace-transformed semi-discrete system (19) can serve as a starting point for the design of time-stepping schemes of various types. Central to the analysis of such methods is the notion of a continuous to discrete mapping. Consider two such mappings:

$$s \rightarrow \zeta_C = \frac{e^{s_d k/2} - e^{-s_d k/2}}{k} \quad s \rightarrow \zeta_B = \frac{2}{k} \frac{e^{s_d k/2} - e^{-s_d k/2}}{e^{s_d k/2} + e^{-s_d k/2}} \quad (23)$$

where here, $s_d = \xi_d + j\omega_d$ is a discrete-time frequency variable, and k is a time step. By sampling considerations, $\omega_d \in (-\pi/k, \pi/k]$. ζ_C corresponds to a centered time difference approximation to a time derivative, and ζ_B to a trapezoid rule approximation. The factor e^{sak} corresponds, in the time domain, to a unit shift in the time variable of k seconds. Thus, under such mappings, the resulting system may be interpreted as a discrete-time recursion. A factor of $e^{sk/2}$ has been extracted from these mappings, which is of no consequence in implementation, in order to simplify the resulting analysis.

A useful fact is that the mapping $s \rightarrow \zeta_B$ is sign-preserving in the real part of the argument. Thus if $\text{Re}(s) > 0$, then $\text{Re}(s_d) = \xi_d > 0$. Another is that under the mapping $s \rightarrow \zeta_B$, the positive realness property is preserved in terms of the discrete frequency variable s_d . That is:

$$\text{Re}(Y(s)) \geq 0 \quad \text{when} \quad \text{Re}(s) \geq 0 \quad \implies \quad \text{Re}(Y(\zeta_B)) \geq 0 \quad \text{when} \quad \text{Re}(s_d) = \xi_d \geq 0 \quad (24)$$

4.1 Fully Implicit Scheme

One very simple and robust approach to scheme design, for a given linear ODE system, is to apply the trapezoid rule uniformly. In terms of the mapping $s \rightarrow \zeta_B$, the system matrix $\mathbf{M}_B(\zeta_B)$ corresponding to the discrete system will be

$$\mathbf{M}_B = \begin{bmatrix} \zeta_B \mathbf{I}_N + \mathbf{B}\mathbf{Y}(\zeta_B)\mathbf{B}^T & -\mathbf{A}^T \\ \mathbf{A} & \zeta_B \mathbf{I}_{N_e} \end{bmatrix} \quad (25)$$

As in the continuous case, it is necessary to ensure that the system matrix does not possess zeros for any discrete frequency s_d^* with $\text{Re}(s_d^*) = \xi_d^* > 0$. The individual entries of the matrix \mathbf{Y} remain positive real under the mapping $s \rightarrow \zeta_B$, and, furthermore, as the mapping preserves the sign of the real part of the frequency variable, the analysis is unchanged from the continuous case. The resulting scheme is thus unconditionally stable.

Though the design is exceedingly simple, such a scheme is not of great practical interest, due to a) the large degree of dispersion it will engender, and b) the extreme computational cost associated with an implicit scheme of this type, rendering the scheme useless for all but very small-scale computations.

4.2 Explicit Scheme

Explicit schemes are of more practical interest, but stability conditions necessarily appear. It is useful to employ a hybridized design, using the centered mapping $s \rightarrow \zeta_C$ over the problem interior, and the trapezoid rule $s \rightarrow \zeta_B$ over the boundary, with the twin goals of a) reducing computational cost and numerical dispersion over the problem interior, and b) allowing for a convenient analysis of numerical boundary termination, given positive real wall admittances.

In this case, the system matrix will be given by

$$\mathbf{M}_{BC} = \begin{bmatrix} \zeta_C \mathbf{I}_N + \mathbf{B}\mathbf{Y}(\zeta_B)\mathbf{B}^T & -\mathbf{A}^T \\ \mathbf{A} & \zeta_C \mathbf{I}_{N_e} \end{bmatrix} \quad (26)$$

As in the continuous case, suppose that \mathbf{M}_{BC} possesses a zero at discrete frequency $s_d^* = \xi_d^* + j\omega_d^*$, for $\xi_d^* > 0$. Given that, for $\xi_d^* > 0$, $\zeta_C(s_d) \neq 0$, this reduces to the condition that the determinant of the Schur complement $\bar{\mathbf{M}}_{BC}(s_d^*)$ of the lower diagonal block vanishes at s_d^* , or

$$\det \left(\underbrace{\zeta_C(s_d^*)\mathbf{I}_N + \mathbf{B}\mathbf{Y}(\zeta_B(s_d^*))\mathbf{B}^T + \frac{1}{\zeta_C(s_d^*)}\mathbf{A}^T\mathbf{A}}_{\bar{\mathbf{M}}_{BC}(s_d^*)} \right) = 0 \quad (27)$$

This implies that there must be a non-zero vector \mathbf{x} such that $x^\dagger \bar{\mathbf{M}}_{BC}(s_d^*)\mathbf{x} = 0$, or

$$\zeta_C(s_d^*)|\mathbf{x}|^2 + \frac{1}{\zeta_C(s_d^*)}|\mathbf{A}\mathbf{x}|^2 + \sum_{b=1}^{N_b} Y_b(\zeta_B(s_d^*))|w_b|^2 = 0 \quad (28)$$

where as before, $\mathbf{w} = \mathbf{B}^T\mathbf{x}$.

First, consider the case of $-\pi/k < \omega_d^* < \pi/k$. Taking the real part of (28) gives

$$\frac{2}{k} \sinh(\xi_d^*k/2) \cos(\omega_d^*k/2) |\mathbf{x}|^2 + \frac{k}{2} \frac{\sinh(\xi_d^*k/2) \cos(\omega_d^*k/2)}{\sinh^2(\xi_d^*k/2) + \sin^2(\omega_d^*k/2)} |\mathbf{A}\mathbf{x}|^2 + \sum_{b=1}^{N_b} \text{Re}(Y_b(\zeta_B(s_d^*))) |w_b|^2 = 0 \quad (29)$$

which cannot be satisfied for $\xi_d^* > 0$, due to the positive realness property of Y_b .

Then, consider the case of $\omega_d^* = \pi/k$. At this frequency, the positive real functions Y_b take on purely real values. Taking the imaginary part of (28) gives

$$\frac{2}{k} \cosh(\xi_d^*k/2) |\mathbf{x}|^2 - \frac{k}{2} \frac{1}{\cosh(\xi_d^*k/2)} |\mathbf{A}\mathbf{x}|^2 = 0 \quad (30)$$

This equation has no solution for $\xi_d^* > 0$ under the condition

$$k \leq \frac{2}{\sqrt{\lambda_{max}(\mathbf{A}^T\mathbf{A})}} \quad (31)$$

where $\lambda_{max}(\mathbf{A}^T\mathbf{A})$ indicates the maximum eigenvalue of $\mathbf{A}^T\mathbf{A}$. This bound reduces to the well-known von Neumann stability condition for typical finite difference schemes [9]. Note that $-\mathbf{A}^T\mathbf{A}$ is essentially a discrete Laplacian operator.

5 Concluding Remarks

The main point of this short contribution is that, given a spatial discretisation of a given room volume, it is possible to reduce the analysis of a resulting time-dependent scheme to an analysis of the problem over the interior, assuming Neumann or perfectly-reflecting conditions. Such analysis requires the evaluation of the spectral radius of the discrete Laplacian operator. Provided that boundary conditions are passive, and that a trapezoid rule is used, the stability condition arrived at through the analysis of the simpler Neumann problem is unchanged.

A flux-conservative formalism has been employed here, resulting in a finite volume method, or, under specialisation to regular grids, a finite difference scheme with a locally-defined stencil. One could begin from other types of constructions for the Laplacian, perhaps non-local, and of higher spatial accuracy; the analysis here under hybrid discretisation rules is not changed, though new constraints on the Laplacian, which must be negative semi-definite. Furthermore, only two examples of discretisations have been discussed here. One could extend the analysis here to cover more elaborate time integration rules over the problem interior, while maintaining the separation of the analysis procedure into interior/boundary analyses.

Locally-reactive conditions have been described here, which are a reasonable approximation to actual wall conditions [10]. This is reflected by the diagonal character of the admittance matrix \mathbf{Y} , for which positive realness, or passivity, is ensured through a positive real condition on each scalar admittance function on the diagonal. The extension to the case of non locally-reactive wall conditions will necessarily imply a non-diagonal admittance matrix \mathbf{Y} , implying inter-cell interaction at the boundary. The positive realness condition need necessarily be extended to that of positive realness for a matrix form, and results presented here will follow as before.

Acknowledgments

This work was supported by the European Research Council, under grant number ERC-StG-2011-279068-NESS.

References

- [1] L. Savioja, T. Rinne, and T. Takala. Simulation of room acoustics with a 3-D finite-difference mesh. In *Proc. Int. Comp. Music Conf.*, pages 463–466, Århus, Denmark, Sept. 1994.
- [2] D. Botteldooren. Acoustical finite-difference time-domain simulation in a quasi-cartesian grid. *J. Acoust. Soc. Am.*, 95(5):2313–2319, 1994.
- [3] S. Bilbao. Modeling of complex geometries and boundary conditions in finite difference/finite volume time domain room acoustics simulation. *IEEE Trans. Audio Speech Language Proc.*, 21(7):1524–1533, 2013.
- [4] S. Bilbao, B. Hamilton, J. Botts, and L. Savioja. Finite volume time domain room acoustics simulation under general impedance boundary conditions. *IEEE/ACM Transactions on Audio Speech and Language Processing*, 24(1):161–173, 2016.
- [5] Allan D Pierce. *Acoustics: an introduction to its physical principles and applications*. McGraw-Hill New York, 1989.
- [6] P. Morse and U. Ingard. *Theoretical Acoustics*. Princeton University Press, Princeton, New Jersey, 1968.
- [7] L. Weinberg. *Network Analysis and Synthesis*. McGraw-Hill, New York, New York, 1962.
- [8] O. Brune. *Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency*. PhD thesis, Massachusetts Institute of Technology, 1931.
- [9] G. E. Forsythe and W. R. Wasow. *Finite-difference methods for partial differential equations*. New York: Wiley, 1960.
- [10] H. Kuttruff. *Room Acoustics*. Taylor and Francis, fourth edition, 2000.