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Optimisation techniques for finite order viscothermal loss modeling in acoustic tubes

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Abstract:

Viscothermal losses for models of one-dimensional wave propagation in ducts are usually expressed, in the frequency domain, in terms of a series impedance/shunt admittance pair. A well-known model is that of Zwikker and Kosten, for which the immittances are not expressed in terms of rational functions of the frequency variable—and thus a difficult match to time domain simulation methods. One approach to finite order rational approximation is based on a high-frequency approximation, leading to a representation in terms of fractional powers of the frequency variable, which can then be further approximated in terms of a rational function using standard techniques. Another is to approximate the Zwikker-Kosten model directly, and a major design consideration is to ensure positive realness under a finite order approximation, leading to a passive or dissipative representation. Though closed-form solutions based on continued fraction expansion are available, another approach is to make use of optimisation over a parameterised finite order rational function for which the positive realness property is inbuilt. This paper focuses on the optimisation problem, using standard iterative techniques such as gradient descent and its extensions, particularly with regard to model order, and extensions to the case of optimisation in a discrete time setting are also discussed. Optimisation results, for a variety of model orders and frequency optimisation ranges, are presented.

Keywords: acoustic tube; finite difference time domain; viscothermal losses; optimisation

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1 Introduction

The modeling of wave propagation in a duct is a problem of longstanding interest. If the duct width is small relative to the wavelengths of interest then a one-dimensional model is appropriate, and serves as a starting point for wind instrument modeling. Standard linear models are usually framed in terms of frequency-domain quantities such as shunt admittance and series impedance, which incorporate viscous and thermal losses in the boundary layer in the tube [1], and such a frequency-domain description may be used immediately in order to arrive at perceptually important characterisations such as the tube input impedance. In some cases, as for, e.g., a cylinder, such an input impedance may be written in closed form; for more general bore profiles, spatial integration strategies (such as the transmission matrix method [2]) may be used in order to generate approximations to the input impedance.

In the interest of developing numerical methods for the case of nonlinear wave propagation, then clearly a frequency domain method will be of limited use, and thus more recent research has focused on time-domain characterisations, for which the shunt admittance and series impedance must be approximated. A usual approach is to make use of a high-frequency approximation to these frequency-domain expressions [3, 4], leading to a PDE system of fractional order in time—one example is the Webster-Lokshin system [5]; see [6] for a comparison of other fractional-order PDE systems under similar approximations. At this point, one may make use of standard discrete approximations to terms of fractional order [7], or, if working in the transmission line context, make use of optimised filter designs [8]. One difficulty here is in the high-frequency approximation itself, leading to a singularity at DC—this is an artefact of the high-frequency approximation and not inherent to the Zwikker-Kosten model, for which immittances are smooth in the limit of low frequency. In this article, optimisation methods are used to find approximations to the Zwikker-Kosten form directly, under the important constraint of passivity. Passive models and optimisation techniques for the continuous-time problem have been discussed recently in [9]. The main contribution of this article is to extend such techniques to perform optimisation directly in the discrete time domain, thus bypassing a major source of error in a simulation setting.

The model of Zwikker and Kosten for viscothermal effects in a cylindrical duct is presented in Section 2, followed by a particular finite order approximation, corresponding to a passive one-port of Foster form, and suitable for optimisation; optimisation results, for this continuous time model are presented. Such a passive representation is transferred to discrete time in Section 3 through the use of the bilinear transformation, leading to frequency warping effects. New optmisation results are presented, taking into account such frequency warping, but maintaining the passivity property of the resulting discrete time approximation. Concluding remarks and perspectives appear in Section 4.

2 Viscous and Thermal Effects in a Cylindrical Duct

Consider a cylindrical duct or tube, of length L, and of radius a. Wave propagation in the duct may be characterised, in the frequency domain, in terms of pressure deviation from atmospheric $P(\omega,x)$, and particle velocity $V(\omega,x)$, provided that the wavelength range is long compared with the tube radius; Here, ω is a an angular frequency, and $x \in [0,L]$ is a spatial coordinate where L is duct length. The coupled pair of differential equations describing the system may be written [2] as

$$\partial_x P + ZV = 0 \qquad\qquad \partial_x V + YP = 0\,. \tag{1}$$

Here, $Z(\omega)$ and $Y(\omega)$ are the series impedance and shunt admittance of the tube and are due to viscous and thermal effects in the tube boundary layer respectively. Forms due to Zwikker and Kosten [1], pertaining to a tube of circular cross section, are:

$$Z = \frac{j\omega\rho}{1-F_v} \qquad Y = \frac{j\omega}{\rho c^2}\left(1+(\gamma-1)F_\theta\right) \tag{2}$$

where, ρ is air density, in kg/m^3, c is the wave speed in m/s, and γ is the ideal gas constant. The functions F_v and F_θ are defined by

$$F_v = \phi\left(\sqrt{-j}r_v\right) \quad F_\theta = \phi\left(\sqrt{-j}r_\theta\right) \quad \phi(\zeta) = \frac{2J_1(\zeta)}{\zeta J_0(\zeta)}\,. \tag{3}$$

where $j = \sqrt{-1}$, J_0 and J_1 are zeroth and first order Bessel functions. The nondimensional viscous/thermal boundary layer thicknesses r_v and r_θ are defined as

$$r_v = \sqrt{S\rho\omega/\pi\eta} \qquad , \qquad r_\theta = \sqrt{S\rho C_p\omega/\pi\kappa}\,, \tag{4}$$

where η is the viscosity coefficient, C_p the specific heat at constant pressure and κ the thermal conductivity. All thermodynamic constants, as well as wave speed and density exhibit a weak dependence on temperature; henceforth, in this article, they are set to values corresponding to a temperature of 26.85°C—see [3] for a table of such values.

2.1 Towards Time-domain Modeling: Further Approximations

The system given in (1) may be used directly in order to compute a frequency domain characterisation of the tube; when complemented by a radiation condition, an exact closed form expression for the input impedance results; the extension to the case of a tube of variable cross section (in which case the system (1) may be extended to the first order form of Webster's equation, and where $Z = Z(\omega,x)$ and $Y = Y(\omega,x)$) may be approximated through a spatial numerical integration for a fixed frequency value. See, e.g., [2].

If, however, one is interested in developing a time domain model, suitable for an extension to the case of nonlinear wave propagation, then further approximations are necessary. One commonly-made approximation is to examine the expressions Z and Y in the high-frequency limit, leading to frequency-domain expressions with fractional power dependence on frequency

[4]. Such frequency domain representations have, of course, an equivalent representation in the time domain, where terms differentiated to fractional order appear—see, e.g., [6]. The Webster-Lokshin model results from further simplification [5].

From the time-dependent forms described above, the transition to discrete time results in another approximation. For differential operators of fractional order, various techniques are available—see, e.g., [7]. Major difficulties are that, first, the approximations to Z and Y are not smooth in the limit of low frequencies, and also that when used in conjunction with a recursive time-stepping method, there is normally no guarantee of numerical stability.

2.2 Passive Representations

The difficulties mentioned above can be sidestepped if one returns to the more general forms of the immittances Z and Y, valid in both the high and low frequency ranges. Indeed, such approximations are smooth in the limit of low frequencies. In what follows, only the viscous series impedance will be considered; the shunt admittance can be treated with very near complete symmetry. It is useful to first decompose Z into a sum of a propagative term and one corresponding purely to viscous effects, written here as Z_v, as

$$Z = j\omega\rho + \underbrace{\frac{j\omega\rho F_v}{1-F_v}}_{Z_v} \tag{5}$$

See Figure 1, showing the real and imaginary parts of Z_v as a function of frequency in Hertz.

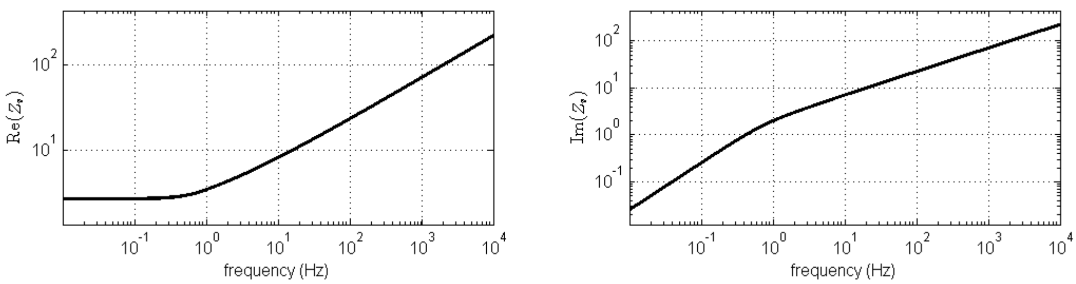

Figure 1: **Real (left) and imaginary (right) parts of Z_v, as a function of frequency in Hz, for a tube of radius 7.5 mm.**

It has been pointed out in [9] that, when extended to the complex plane, Z_v possesses an infinite series of poles and zeros which interlace on the negative real axis, and thus may be viewed as a positive real immittance, which may be realised in terms of elements of two types (resistances and inductances). Among the canonical forms for such immittances are those of Cauer type, corresponding to continued fraction expansion, and presented recently in the present case by Thompson et al. [10]. Foster-type structures [11] are also available, and, as mentioned in [9], the additive structure of the Foster decomposition lends itself to optimisation procedures. When truncated to Mth order, an approximation to Z_v^M to Z_v may be written as

$$Z_v^M = R_0 + \sum_{q=1}^{M} \frac{R_q L_q j\omega}{R_q + L_q j\omega} \tag{6}$$

for some parameters $R_q \geq 0,\ q = 0,\ldots,M$, and $L_q \geq 0,\ q = 1,\ldots,M$. Provided the non-negativity conditions on these parameters are satisfied, the immittance remains positive real, or passive, under any order of truncation. Though in principle, it is possible to set the pole/zero structure of Z_v^M to match that of Z_v up to a given order, in practice it is more useful to retain this structure as a starting point for optimisation procedures.

2.3 Optimisation

There is great design flexibility using the finite order model given in (12). In this case, the objective function chosen will be

$$E = \sum_{k=1}^{N} \left| \frac{Z_v^M(\omega_k) - Z_v(\omega_k)}{Z_v(\omega_k)} \right|^2 \tag{7}$$

over a set of N frequencies $\omega_k,\ k = 1,\ldots,N$ covering a range of interest (which does not necessarily include the DC frequency). In order to ensure that the approximation remains passive through the optimisation procedure, it is useful to rewrite Z_v^M as

$$Z_v^M = e^{b_0} + \sum_{q=1}^{M} \frac{e^{b_q} j\omega}{e^{a_q} + j\omega} \tag{8}$$

in terms of the real parameters $a_q,\ q = 1,\ldots,M$ and $b_q,\ q = 0,\ldots,M$, which will be used directly in the optimisation procedure.

As is well known, using standard gradient descent can lead to extremely slow convergence, and this is indeed the case here; much better is Newton's method, requiring the calculation of the Hessian of the objective function from (7); again, due to the simple additive form of (8), this is calculated in a straightforward manner. Optimisation results, for the particular choice of Z_v as illustrated in Figure 1, for orders $M = 4$, $M = 8$ and $M = 16$ are shown in Figure 2, for both a wide frequency range (0.1 Hz to 10000 Hz), and for a narrow range corresponding to the musical range in wind instruments (20 Hz to 3000 Hz). These results are noticeably superior to those presently previously in [9]. Notice that these fits take into account the reactive behaviour of Z_v; if one is only interested in loss modeling, one can do even better if one optimises over the real part of Z_v alone.

3 Discrete Time

In a discrete-time setting, approximations to the viscothermal immittances are necessary. This is true across a variety of methodologies, including, e.g., digital waveguide models [8], where a consolidated filter representing loss over the entire bore length is used, and also in the case of finite difference time domain methods, where such losses are approximated locally [6].

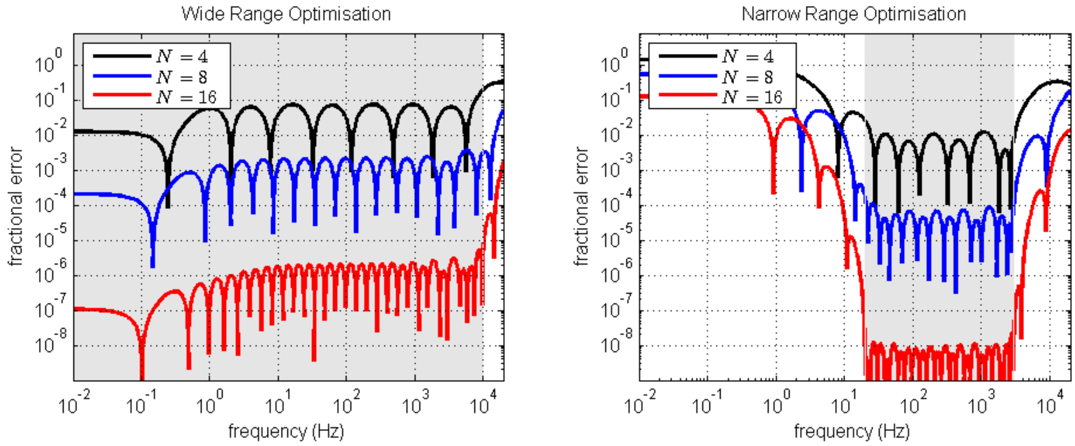

Figure 2: **Fractional error in the real part of the approximation of Z_v^M to Z_v, as a function of frequency in Hz, for different orders M of approximation. Left: error over a wide frequency range of 0 Hz to 10000 Hz, and right, error over a narrow range of frequencies 20 Hz to 3000 Hz.**

The Foster structure presented in (12) above represents a passive approximation to the impedance Z_v, for any real values of the parameters b_q, $q = 0, \ldots, M$ and a_q, $q = 1, \ldots, M$. It is useful to preserve this passivity property in the discrete-time equivalent. A well-known discretisation rule which preserves the passivity property in discrete time is the trapezoid rule, which, in the frequency domain, may be written as the mapping

$$s \longrightarrow \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \tag{9}$$

where $z = e^{sT}$ represents a unit advance of T seconds; such a discretisation rule, when transferred to a discrete time algorithm, leads to computable structures at a sample rate of $1/T$. A positive real function, corresponding to a passive immittance, is mapped to a function which is positive real in the outer disk [12]. See [13] for a description of the use of such viscothermal loss filters within a complete simulation of an acoustic tube.

3.1 Frequency Warping

One obvious difficulty with the optimisation procedure described above is that it does not take into account frequency warping due to the mapping (9). That is, for real frequencies ω, where $\omega = \mathrm{Im}(s)$, the mapping (9) can be written as

$$\omega \longrightarrow \tilde{\omega} = \frac{2}{T} \tan(\omega T/2) \tag{10}$$

The optimised impedances $Z_v(\omega)$ are thus mapped to the discrete-time impedances $Z_v(\tilde{\omega})$, which are no longer a good match to the reference impedance Z_v, and increasingly so with filter order M. The effects of warping can be seen, even at relatively high sample rates. See

Figure 3 at left, illustrating the effects of warping on the error in the impedance approximation, for a narrow-range optimisation, when operating at a sample rate of 100 kHz.

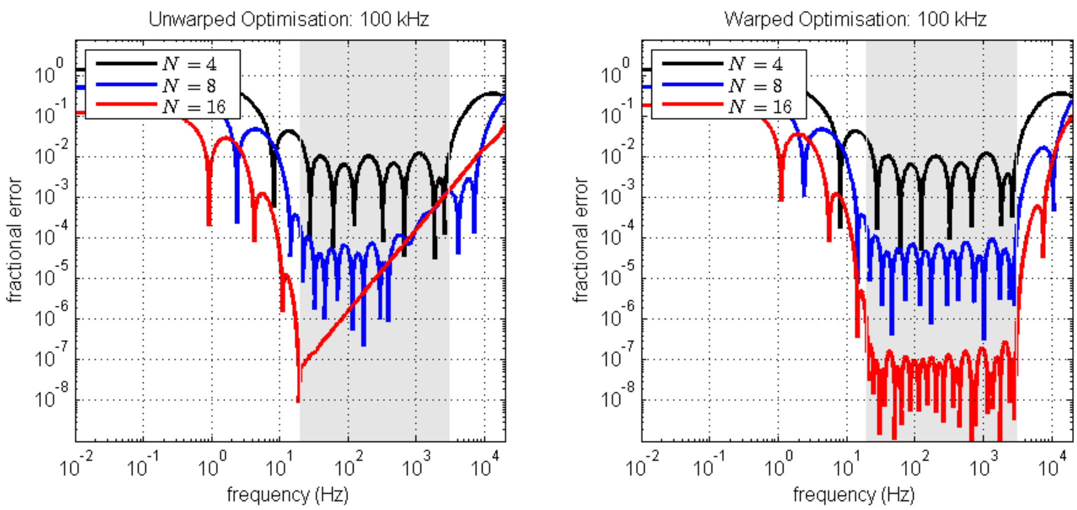

Figure 3: **Fractional error in the real part of the discrete-time approximation of $Z_v^M(\tilde{\omega})$ to $Z_v(\omega)$, as a function of frequency in Hz, for different orders M of approximation, and over a narrow range of frequencies 20 Hz to 3000 Hz. The sample rate is 100 kHz. Left: error when continuous-time optimised design parameters are used, and right: under frequency-warped optimisation.**

The obvious approach to optimisation, then, is to build such warping directly into the cost function, which may be redefined as

$$E = \sum_{k=1}^{N} \left| \frac{Z_v^M(\tilde{\omega}_k) - Z_v(\omega_k)}{Z_v(\omega_k)} \right|^2 \tag{11}$$

again over a set of N frequencies ω_k, $k = 1, \ldots, N$ covering a range of interest, with $\tilde{\omega}_k = (2/T)\tan(\omega_k T/2)$, where $Z_v^M(\tilde{\omega})$ is defined as

$$Z_v^M(\tilde{\omega}) = e^{b_0} + \sum_{q=1}^{M} \frac{e^{b_q} j\tilde{\omega}}{e^{a_q} + j\tilde{\omega}} \tag{12}$$

Results are clearly superior using optimisation over warped frequencies—see Figure 3 at right.

One implication is that designs parameters must be obtained separately through optimisation for each target sample rate. The results differ, but in all cases, results are noticeably improved when an optimisation is performed taking into account warping effects. See Figure 4, illustrating approximation errors at a variety of sample rates.

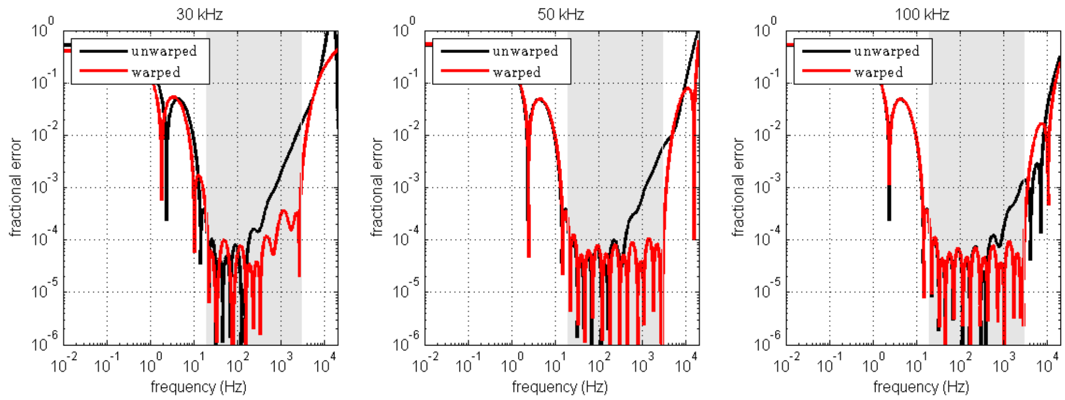

Figure 4: **Fractional error in the real part of the discrete-time approximation of $Z_v^M(\tilde{\omega})$ to $Z_v(\omega)$, as a function of frequency in Hz, for $M = 8$, and over a narrow range of frequencies 20 Hz to 3000 Hz, for both warped and unwarped designs. Sample rates are 30 kHz (left), 50 kHz (center) and 100 kHz (right).**

4 Concluding Remarks

In this short paper, passive discrete-time approximations to viscothermal immittances have been presented. This extends previous work on optimised designs in the continuous time case in [9]. The main result is that through a particular choice of immittance (namely the Foster form), passivity may be ensured by maintaining non-negativity constraints on the immittance parameters through the optimisation procedure. This may be extended to the fully discrete case in a passivity-preserving discretisation rule is employed—in this case, the trapezoid rule or bilinear transformation. In addition, the optimisation may be performed directly in the discrete time domain, bypassing any frequency warping effects.

Such optimised designs, though they can be extremely accurate, can pose some difficulties is used in conjunction with a time domaiin numerical method. One is that optimisation must be performed anew for each distinct sample rate, for best results. Another more general problem is that in the case of tubes of variable cross section, a distinct optimised design should be obtained as a function of bore radius. Thus there may be some advantage in terms of algorithm simplicity, in using closed form designs, such as the Cauer structure proposed in [10], though higher order designs will be necessary in practice, particularly in musical applications.

Acknowledgments

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References

[1] C. Zwikker and C. Kosten. *Sound Absorbing Materials*, pages 1–147. Elsevier, New York, 1949.

[2] R. Caussé, J. Kergomard, and X. Lurton. Input impedance of brass musical instruments—comparison

between experiment and numerical models. *J. Acoust. Soc. Am.*, 75(1):241–254, 1984.

[3] A. Benade. On the propagation of sound waves in a cylindrical conduit. *J. Acoust. Soc. Am.*, 44(2):616–623, 1968.

[4] D. Keefe. Acoustical wave propagation in cylindrical ducts: Transmission line parameter approximations for isothermal and nonisothermal boundary conditions. *J. Acoust. Soc. Am.*, 75(1):58–62, 1984.

[5] R. Mignot, T. Hélie, and D. Matignon. Digital waveguide modeling for wind instruments: Building a state space representation based on the Webster Lokshin model. *IEEE Trans. Audio, Speech, Language Proc.*, 18(4):843–854, 2010.

[6] S. Bilbao and J. Chick. Finite difference time domain simulation for the brass instrument bore. *J. Acoust. Soc. Am.*, 134(5):3860–3871, 2013.

[7] Y. Chen and K. Moore. Discretization schemes for fractional-order differentiators and integrators. *IEEE Transactions on Circuits andSystems-I: Fundamental Theory and Applications*, 49(3):363–367, 2002.

[8] J. Abel, T. Smyth, and J. O. Smith III. A simple, accurate wall loss filter for acoustic tubes. In *Proceedings of the 6th International Digital Audio Effects Conference*, pages 254–258, London, UK, September 2003.

[9] S. Bilbao, R. Harrison, J. Kergomard, B. Lombard, and C. Vergez. Passive models of viscothermal wave propagation in acoustic tubes. *J. Acoust. Soc. Am.*, 138(2):555–558, 2015.

[10] S. Thompson, T. Gabrielson, and D. Warren. Analog model for thermoviscous propagation in a cylindrical tube. *J. Acoust. Soc. Am.*, 135(2):585–590, 2014.

[11] L. Weinberg. *Network Analysis and Synthesis*, page 312. R. E. Kreiger, New York, New York, 1975.

[12] J. O. Smith III. *Techniques for Digital Filter Design and System Identification with Application to the Violin*. PhD thesis, Department of Electrical Engineering, Stanford University, 1983.

[13] S. Bilbao and R. Harrison. Passive time-domain numerical models of viscothermal wave propagation in acoustic tubes of variable cross section. *J. Acoust. Soc. Am.*, 2016.